THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A
Solution to Homework 8

Compulsory Part

Sec. 6.1

(Sec 6.1 Q11) Q: Prove the parallelogram law on an inner product space V; that is show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \text{ for all } x, y \in V$$

What does this equation state about paralleograms in $\mathbb{R}^2$?

Ans :

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2 \\ &= 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

(Sec 6.1 Q12) Q: Let $\{v_1, \cdots, v_k\}$ be and orthogonal set, $\{a_1, \cdots, a_k\}$ be scalars. Prove that

$$\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

Ans:

$$\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{i=1}^{k} a_i v_i \right\rangle = \sum_{i,j=1}^{k} a_i a_j \langle v_i, v_j\rangle = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2, \tag{1}$$

since if $i \neq j$, $\langle v_i, v_j\rangle = 0$.

(Sec 6.1 Q17) Q: Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one to one.

Ans: Let $x \in V$ such that $T(x) = 0$, then $\|x\| = \|T(x)\| = 0$, hence $x = 0$, so the kernel is $\{0\}$ hence T is one to one.

(Sec 6.1 Q18) Let V be a vector space over F, where $F = \mathbb{R}$ or $F = \mathbb{C}$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot\rangle$. If $T : V \to W$ is linear, prove that $\langle x, y\rangle' = \langle T(x), T(y)\rangle$ defines an inner product on V if and only if T is one to one.

Ans :
($\Rightarrow$) : Suppose $T(x) = T(y)$. We consider

$$\begin{aligned}\langle x-y, x-y\rangle' &= \langle x, x\rangle' - 2Re\langle x, y\rangle' + \langle y, y\rangle' \\ &= \|T(x)\|^2 - 2\|T(x)\|^2 + \|T(y)\|^2 = 0.\end{aligned}$$

Hence $y = x$.
$(\Leftarrow)$: Linearity is trivial. Also it is trivial that $\overline{\langle x, y\rangle'} = \langle y, x\rangle'$. We now show that $\langle x, x\rangle' > 0$ if $x \neq 0$. Since T is one to one hence its kernel is $\{0\}$. Hence $\langle x, x\rangle' = \langle T(x), T(x)\rangle = \|T(x)\| > 0$ since $T(x) \neq 0$, and $\langle \cdot, \cdot\rangle$ is an inner product.

(Sec 6.1 Q22) Let β be a basis forV. $x = \sum_{i=1}^{n} a_i v_i$, $y = \sum_{i=1}^{n} b_i v_i$, define $\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b_i}$.

(a) Prove $\langle \cdot, \cdot\rangle$ is an inner product.

(b) Prove if $V = R^n$ or C^n and β is the standard basis, then the definition above is the standard inner product.

(a) Direct checking.

(b) Let $(\cdot, \cdot)$ denote the standard inner product. If $x = (a_1, \cdots, a_n)$, $y = (b_1, \cdots, b_n)$, then $\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b_i} = (x, y)$.

Optional Part

Sec. 6.1

(Sec 6.1 Q01) Ans:

(a) T.

(b) T.

(c) F. Not in the second component.

(d) F. Consider 6.1 Q18.

(e) F.

(f) F.

(g) F.

(h) T.

(Sec 6.1 Q08) Ans:

(a) Let $a = b = c = d = 1$, $x = (1, 1) \neq 0$, then $\langle x, x\rangle = 0$.

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A \neq 0$, $\langle A, A\rangle = 0$.

(c) Let $f = 1 \neq 0$ Then $\langle f, f\rangle = 0$

(Sec 6.1 Q19) Ans:

(a)

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x + y\rangle + \langle y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 \\
&= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2.
\end{aligned}$$

(b)

$$\|x\| \leq \|x-y\| + \|y\|$$
$$\|y\| \leq \|x-y\| + \|x\|$$

Together we have the desired inequality.

(Sec 6.1 Q20) Ans :

(a) Direct checking.

(b) Direct checking.

(Sec 6.1 Q21) Ans :

(a) Direct checking.

(b)

$$\begin{aligned} A_1 + iA_2 &= B_1 + iB_2 \\ A_1 - B_1 &= i(A_2 - B_2) \\ (A_1 - B_1)^* &= (i(A_2 - B_2)))^* \\ A_1 - B_1 &= -i(A_2 - B_2) \end{aligned}$$

Hence

$$\begin{aligned} A_1 - B_1 &= -(A_1 - B_1) \\ A_1 - B_1 &= 0 \\ A_1 &= B_1. \end{aligned}$$

and hence $A_2 = B_2$.

(Sec 6.1 Q23) Ans:

(a)

$$\begin{aligned} \langle x, Ay \rangle &= \sum_{i=1}^{n} x_i \overline{(Ay)_i} \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \overline{A_{ij} y_j} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} A^*_{ji} x_i \overline{y_j} \\ &= \langle A^* x, y \rangle. \end{aligned}$$

(b)

$$\langle x, Ay \rangle = \langle A^* x, y \rangle = \langle Bx, y \rangle \implies \langle A^* x - Bx, y \rangle = 0$$

Hence $A^* = B$.

(c)

$$\begin{aligned}
(Q^*Qx)_i &= \sum_{j=1}^{n} (Q^*Q)_{ji} x_i \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} Q^*_{jk} Q_{ki} x_i \\
&= \sum_{j=1}^{n} \langle Q_i, Qj \rangle x_i \\
&= \sum_{j=1}^{n} \delta_{ij} x_i = x_i
\end{aligned}$$

(d) Let β' be the standard basis, then we have

$$\begin{aligned}
[T]_\beta &= [I]_{\beta'}^{\beta} [T]_{\beta'} [I]_{\beta}^{\beta'} \\
&= Q^* A Q,
\end{aligned}$$

So

$$[T]_\beta^* = Q^* A^* Q = [IUI]_\beta = [U]_\beta.$$